Lecture #1: Nuclear and Thermonuclear Reactions

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THE UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Nuclear Reactions

Definition of cross section:

$$\sigma \equiv \frac{\left(\text{number of interactions per time}\right)}{\left(\text{number of incident particles per area per time}\right)\left(\text{number of target nuclei within the beam}\right)} = \frac{N_r}{N_0 N_t}$$

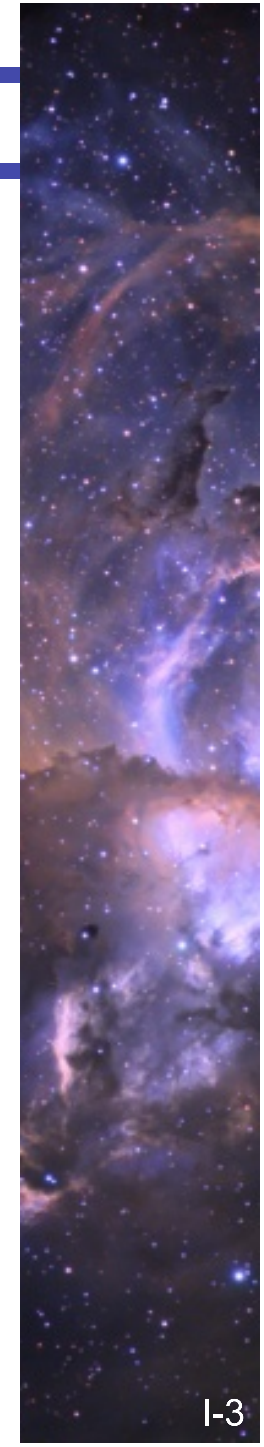
Unit: 1 barn=10^{-28} m^2

Example: $^1H + {}^1H \rightarrow {}^2H + e^+ + \nu$ (first step of pp chain)

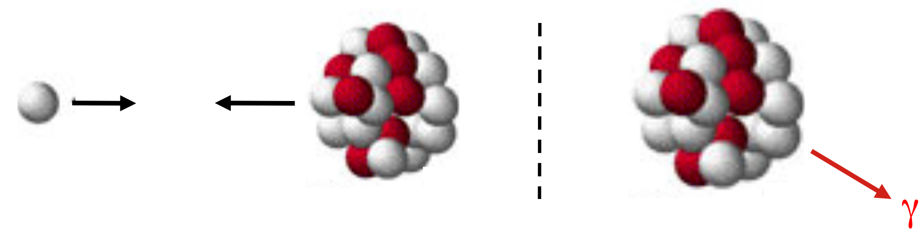
$\sigma_{theo}=8 \times 10^{-48}$ cm^2 at E_{lab}=1 MeV [E_{cm}=0.5 MeV]

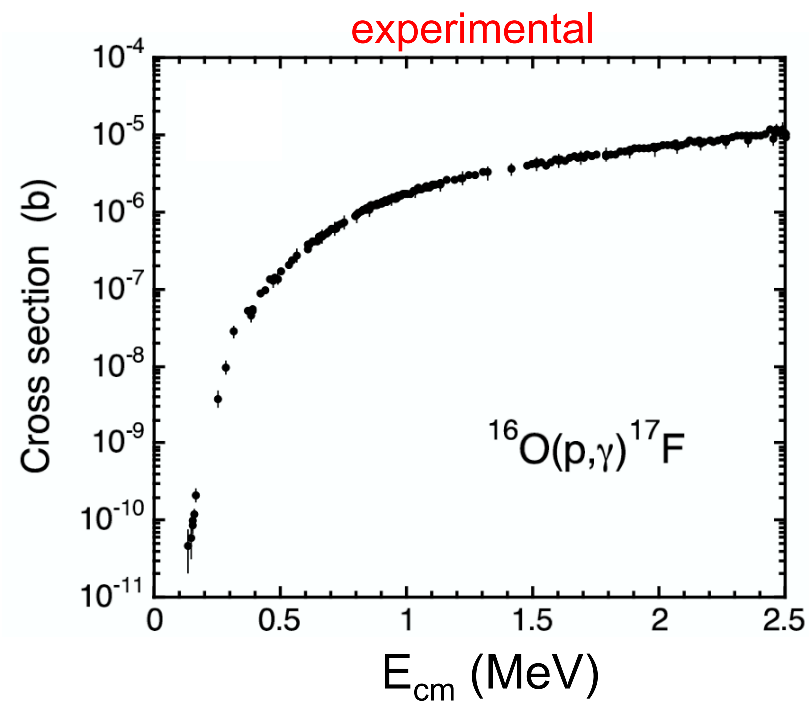
1 ampere (A) proton beam (6×10^{18} p/s) on dense proton target (10^{20} p/cm^2)

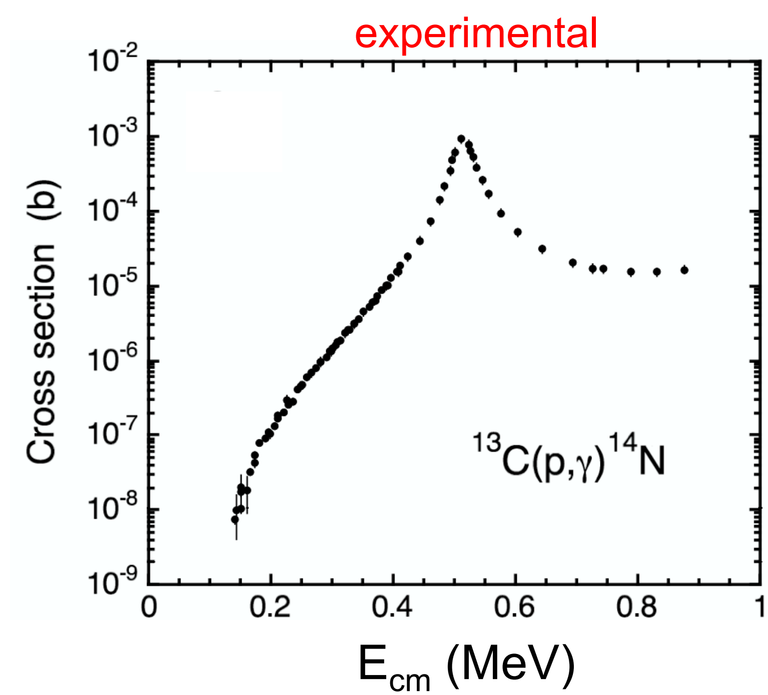
gives only 1 reaction in 6 years of measurement!



Cross Sections

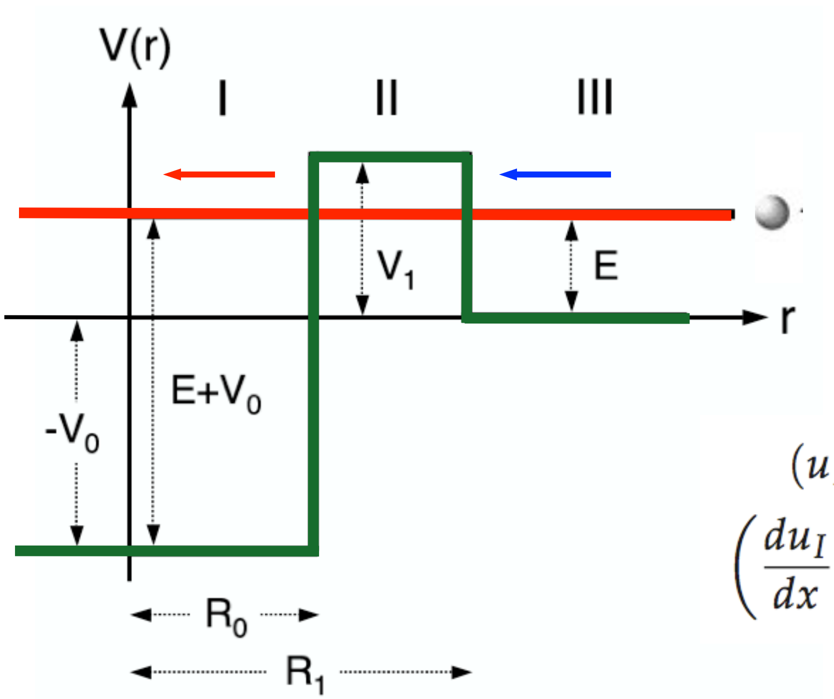






(i) why does the cross section fall drastically at low energies?
(ii) where is the peak in the cross section coming from?

A Simple Example in 1 Dimension



Wave function solutions:

$$u_I = Ae^{iKx} + Be^{-iKx}$$
$$u_{II} = Ce^{-\kappa x} + De^{\kappa x}$$
$$u_{III} = Fe^{ikx} + Ge^{-ikx}$$

$$K^2 = \frac{2m}{\hbar^2}(E + V_0)$$
$$\kappa^2 = \frac{2m}{\hbar^2}(V_1 - E)$$
$$k^2 = \frac{2m}{\hbar^2}E$$

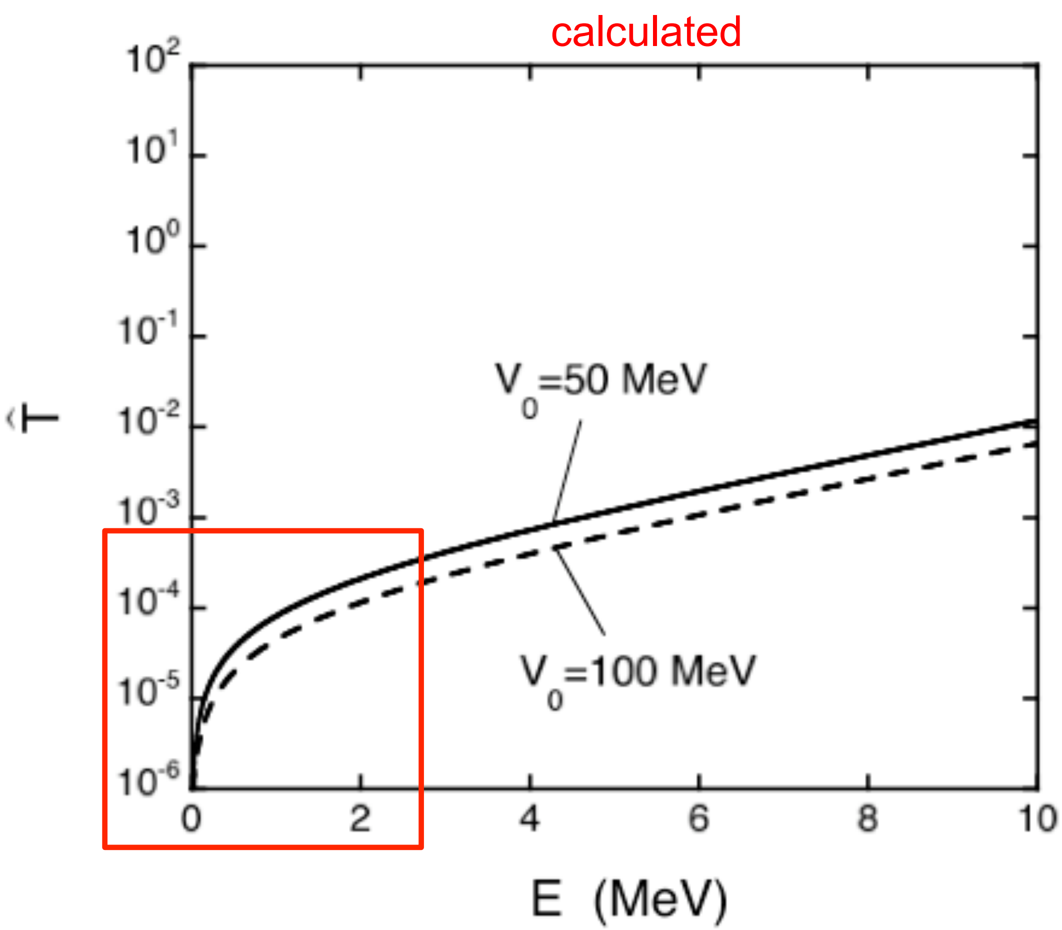
Continuity condition:

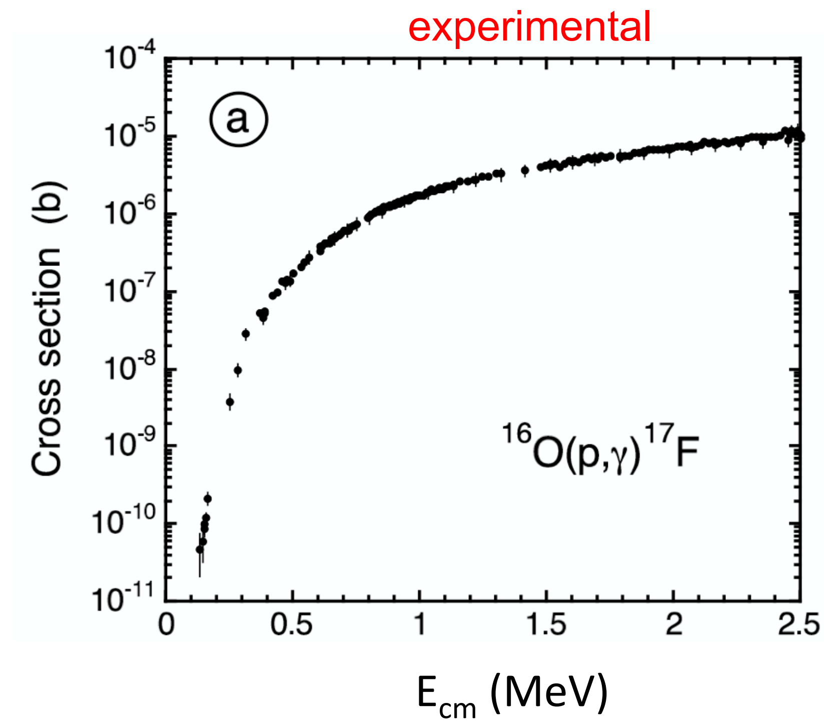
$$(u_I)_{R_0} = (u_{II})_{R_0} \qquad (u_{II})_{R_1} = (u_{III})_{R_1}$$
$$\left(\frac{du_I}{dx}\right)_{R_0} = \left(\frac{du_{II}}{dx}\right)_{R_0} \qquad \left(\frac{du_{II}}{dx}\right)_{R_1} = \left(\frac{du_{III}}{dx}\right)_{R_1}$$

Transmission coefficient: $\hat{T} = \frac{K}{k}\frac{|B|^2}{|G|^2} \approx e^{-(2/\hbar)\sqrt{2m(V_1-E)}(R_1-R_0)}$

(after lengthy algebra, and for the limit of low E)

"Tunnel effect"

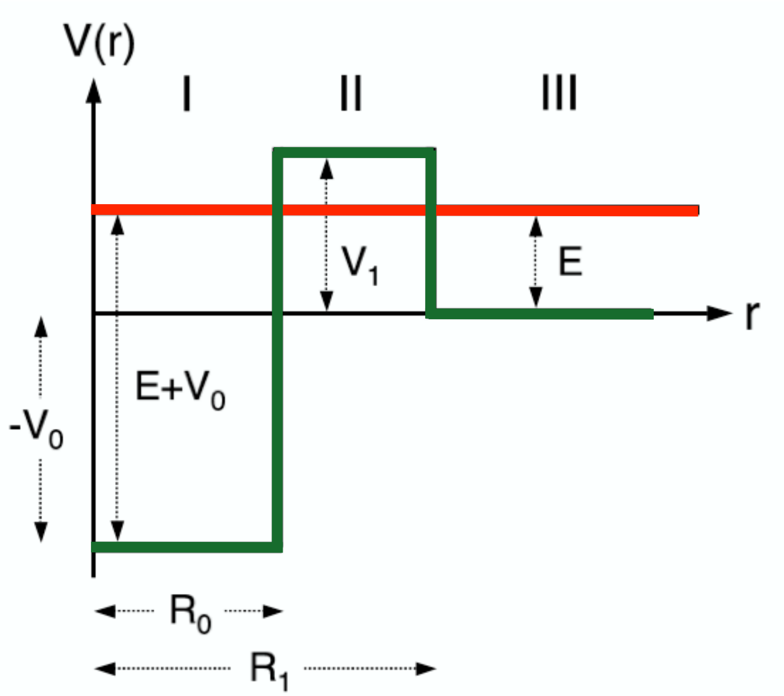




Tunnel effect is the reason for the strong drop in cross section at low energies!

Back to the Simple Potential, Now in 3 Dimensions

$$\lambda = \frac{2\pi}{K}$$



wave function solutions:

$$u_I = A' \sin(Kr)$$
$$u_{II} = Ce^{-\kappa r} + De^{\kappa r}$$
$$u_{III} = F' \sin(kr + \delta_0)$$

$$K^2 = \frac{2m}{\hbar^2}(E + V_0)$$
$$\kappa^2 = \frac{2m}{\hbar^2}(V_1 - E)$$
$$k^2 = \frac{2m}{\hbar^2}E$$

Continuity condition…

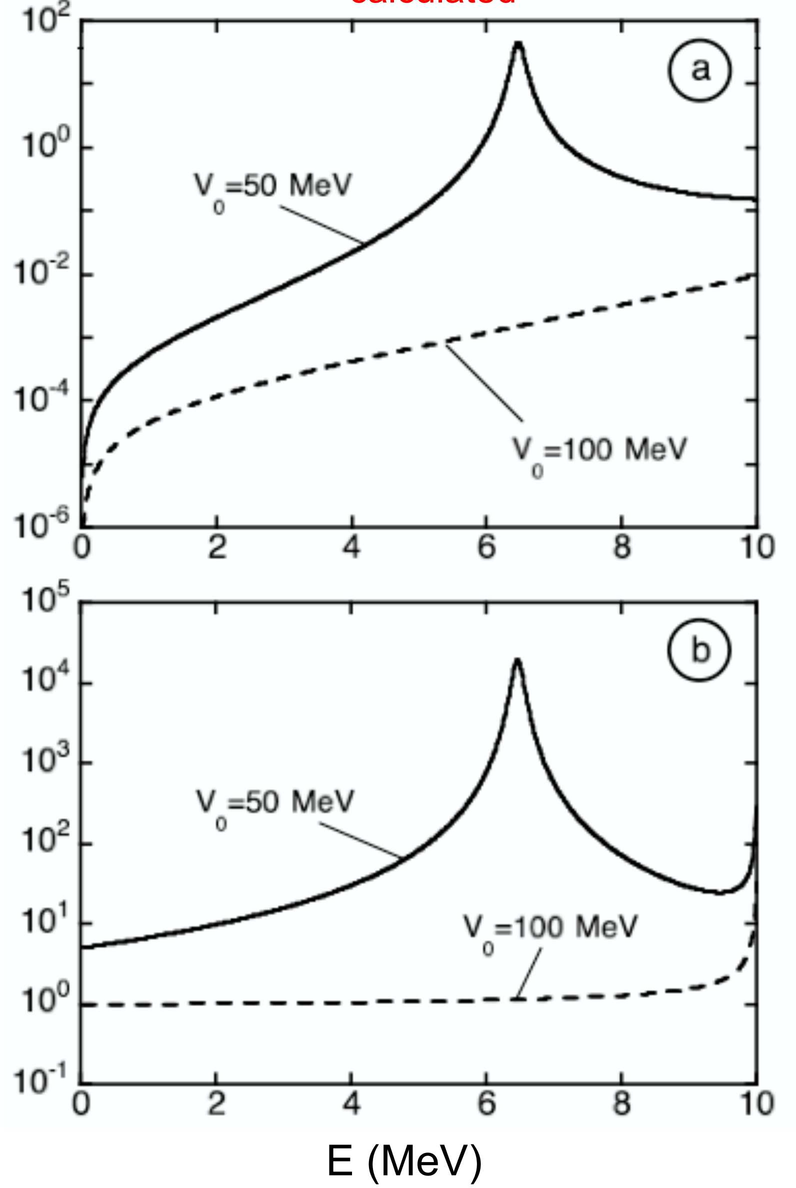
Wave intensity in interior region:
(after very tedious algebra)

$$\frac{|A'|^2}{|F'|^2} = \left\{ \sin^2(KR_0) + \left(\frac{K}{k}\right)^2 \cos^2(KR_0) + \sin^2(KR_0)\sinh^2(\kappa\Delta)\left[1 + \left(\frac{\kappa}{k}\right)^2\right] + \cos^2(KR_0)\sinh^2(\kappa\Delta)\left[\left(\frac{K}{\kappa}\right)^2 + \left(\frac{K}{k}\right)^2\right] \right.$$
$$\left. + \sin(KR_0)\cos(KR_0)\sinh(2\kappa\Delta)\left[\left(\frac{K}{\kappa}\right) + \left(\frac{K}{\kappa}\right)\left(\frac{\kappa}{k}\right)^2\right] \right\}^{-1}$$

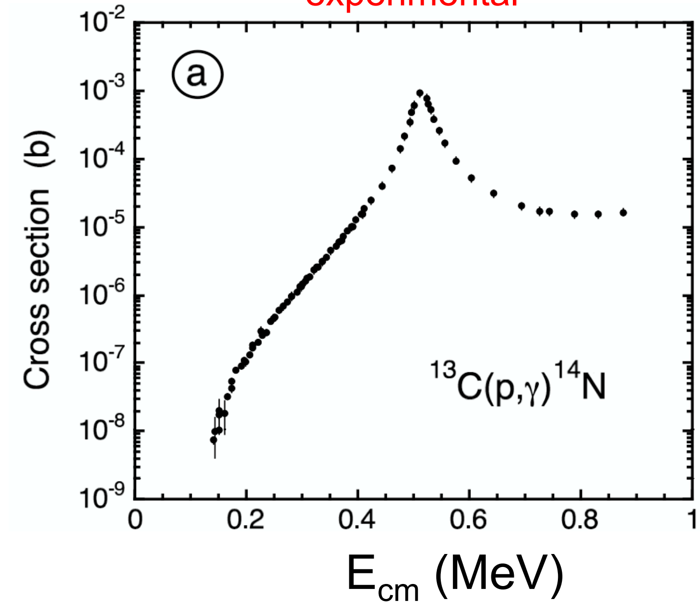
calculated

$$\frac{|A'|^2}{|F'|^2}$$

$$\frac{|A'|^2}{|F'|^2}\,\frac{1}{\hat{T}}$$



experimental



[change of potential depth V_0:
changes wavelength in interior region]

"Resonance phenomenon"

A resonance results from favorable wave function matching conditions at the boundaries!

Transmission Through the Coulomb Barrier

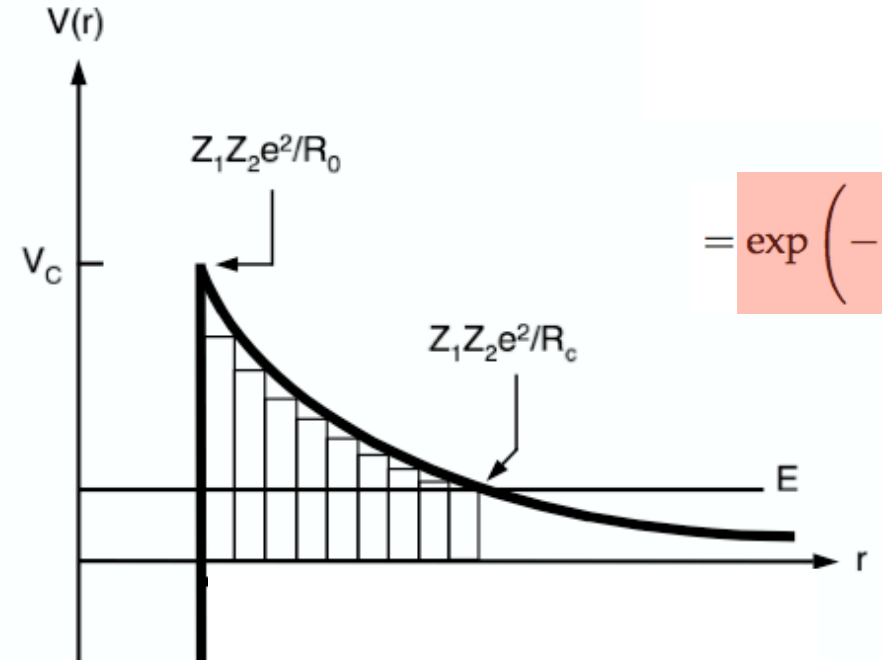
$$\hat{T} = \hat{T}_1 \cdot \hat{T}_2 \cdot \ldots \cdot \hat{T}_n \approx \exp\left[-\frac{2}{\hbar}\sum_i \sqrt{2m(V_i - E)}(R_{i+1} - R_i)\right]$$

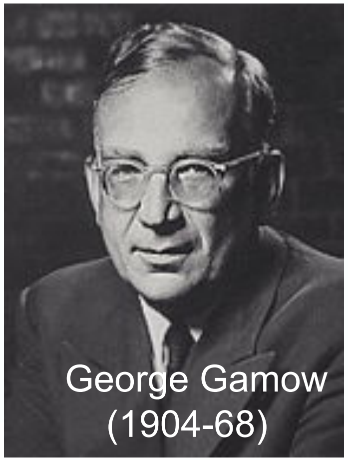
$$\xrightarrow[n \text{ large}]{} \exp\left[-\frac{2}{\hbar}\int_{R_0}^{R_c} \sqrt{2m[V(r) - E]}\, dr\right]$$

$$= \exp\left(-\frac{2\pi}{\hbar}\sqrt{\frac{m}{2E}} Z_0 Z_1 e^2 \left[1 + \frac{2}{3\pi}\left(\frac{E}{V_C}\right)^{3/2}\right] + \frac{4}{\hbar}\sqrt{2mZ_0Z_1e^2R_0}\right)$$

[for low energies and zero angular momentum]

“Gamow factor” $e^{-2\pi\eta}$





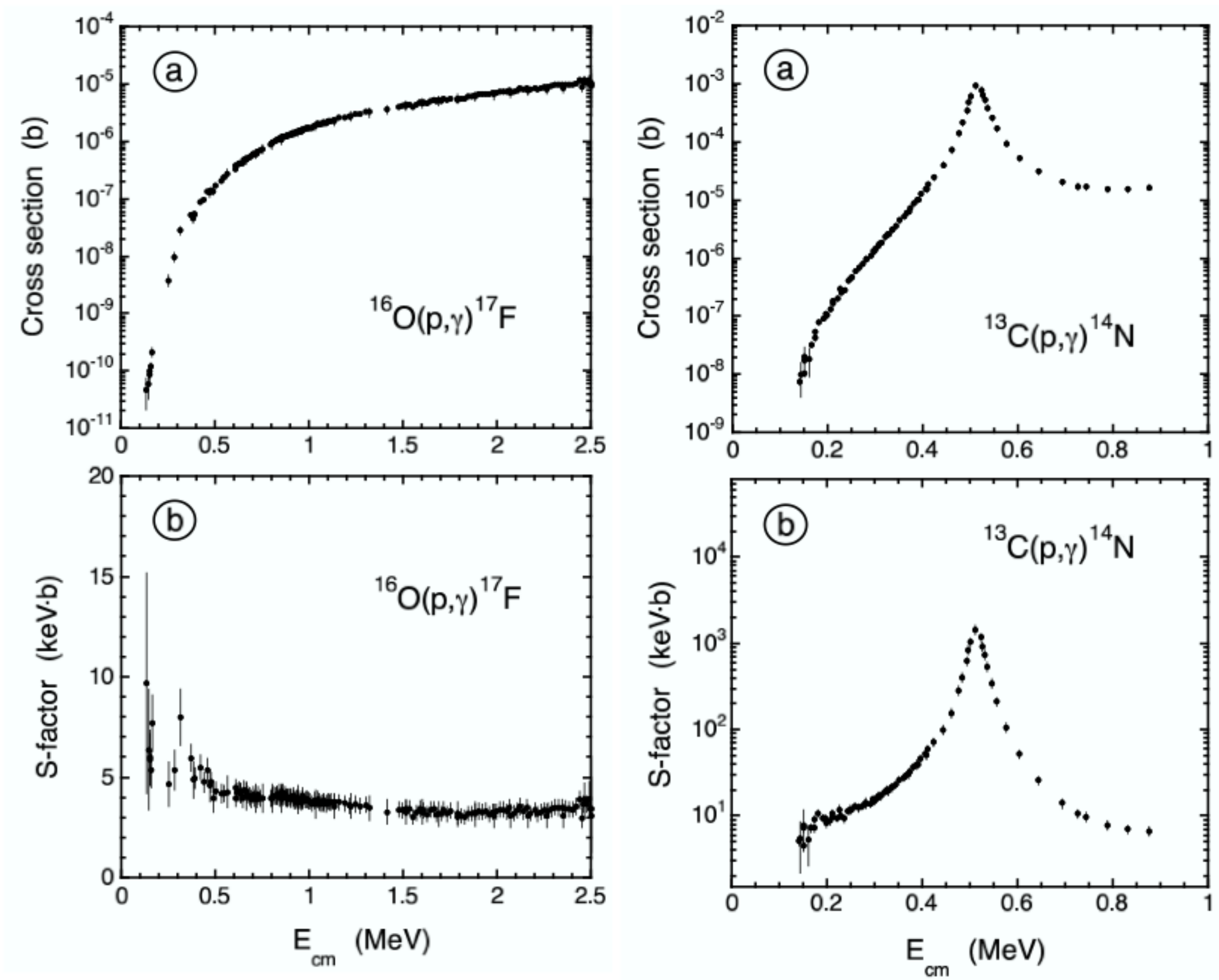
$$\sigma(E) \equiv \frac{1}{E} e^{-2\pi\eta} S(E)$$

“astrophysical S-factor”

Comparison: S-Factors and Cross Sections

cross sections

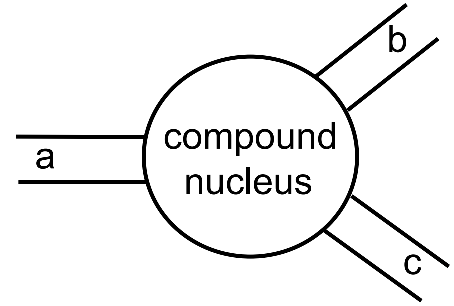
S-factors



Formal Reaction Theory: Breit-Wigner Formula

Eugene Wigner
(1902-95)
Nobel Prize 1963

Example: ^{22}Ne(α,γ)^{26}Mg
^{22}Ne(α,n)^{25}Mg



de Broglie wavelength

partial widths for incoming and outgoing channel

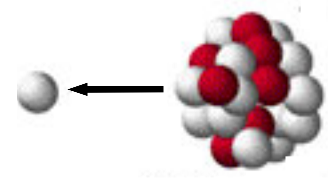
$$\sigma_{\mathrm{BW}}(E) = \frac{\lambda^2}{4\pi} \underbrace{\frac{(2J+1)(1+\delta_{01})}{(2j_0+1)(2j_1+1)}}_{\text{spin factor}} \frac{\Gamma_a \Gamma_b}{(E_r - E)^2 + \Gamma^2/4}$$

resonance energy

total width

Used for: - for fitting data to deduce resonance properties
- for “narrow-resonance” thermonuclear reaction rates
- for extrapolating cross sections when no measurements exist
- for experimental yields when resonance cannot be resolved

What are "Partial Widths"?



probability per unit time for formation or decay of a resonance (in energy units)

For protons/neutrons:

$$\boxed{\Gamma_{\lambda c} = 2\gamma_{\lambda c}^2 P_c = 2\frac{\hbar^2}{mR^2}\, C^2S\ \theta_{pc}^2\ P_c}$$

A partial width can be factored into 3 probabilities:

C^2S: probability that nucleons will arrange themselves in a "residual nucleus + single particle" configuration ["spectroscopic factor"]

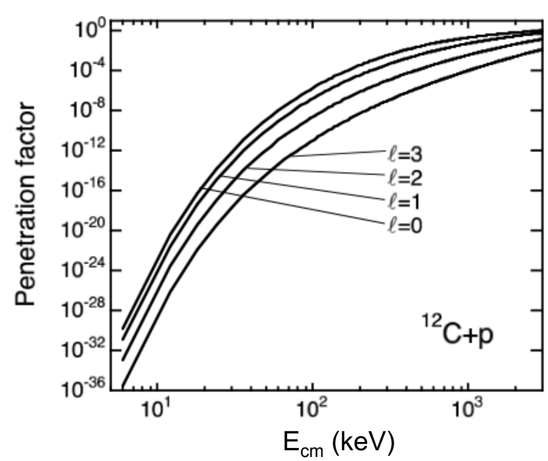
θ^2: probability that single nucleon will appear on nuclear boundary ["dimensionless reduced single particle width"; Iliadis, Nucl. Phys. A 618, 166 (1997)]

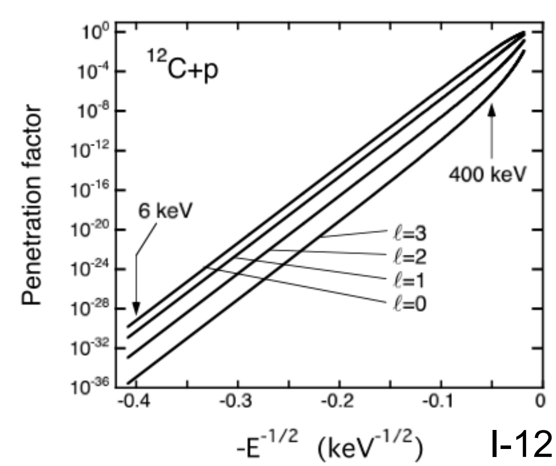
P_c: probability that single nucleon will penetrate Coulomb and centripetal barriers ["penetration factor"]

strongly energy-dependent:

$$P_\ell = R\left(\frac{k}{F_\ell^2 + G_\ell^2}\right)_{r=R}$$

$$\propto e^{-2\pi\eta} = e^{-const/\sqrt{E}}$$





Thermonuclear Reactions

For a reaction 0 + 1 → 2 + 3 we find from the definition of σ (see earlier) a "reaction rate":

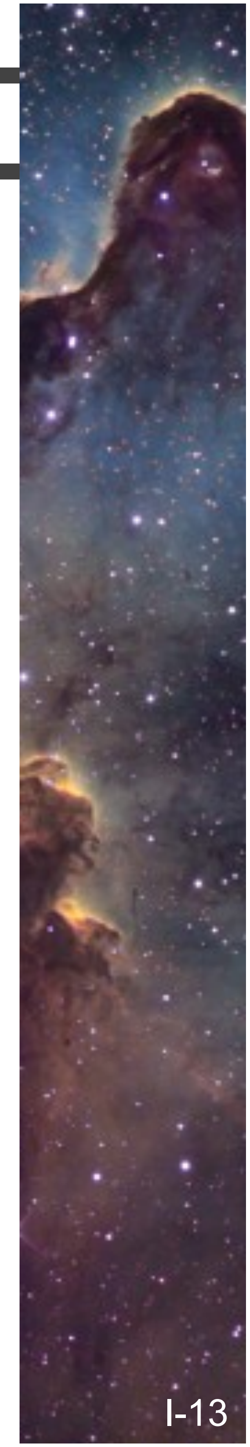
$$r_{01} = N_0 N_1 \int_0^\infty v P(v) \sigma(v) \, dv \equiv N_0 N_1 \langle \sigma v \rangle_{01}$$

For a stellar plasma: kinetic energy for reaction derives from thermal motion:

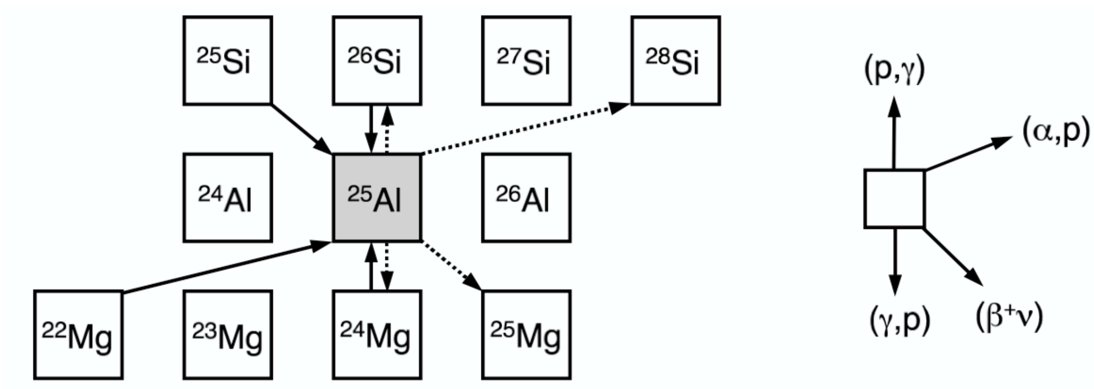
"Thermonuclear reaction"

For a Maxwell-Boltzmann distribution:

$$\boxed{\langle \sigma v \rangle_{01} = \left(\frac{8}{\pi m_{01}} \right)^{1/2} \frac{1}{(kT)^{3/2}} \int_0^\infty E \, \sigma(E) \, e^{-E/kT} \, dE}$$



Interplay of Many Different Nuclear Reactions in a Stellar Plasma



$$\begin{aligned}
\frac{d(N_{^{25}\mathrm{Al}})}{dt} = & \left.\begin{aligned} & N_{\mathrm{H}} N_{^{24}\mathrm{Mg}} \langle \sigma v \rangle_{^{24}\mathrm{Mg}(\mathrm{p},\gamma)} + N_{^{4}\mathrm{He}} N_{^{22}\mathrm{Mg}} \langle \sigma v \rangle_{^{22}\mathrm{Mg}(\alpha,\mathrm{p})} \\ & + N_{^{25}\mathrm{Si}} \lambda_{^{25}\mathrm{Si}(\beta^{+}\nu)} + N_{^{26}\mathrm{Si}} \lambda_{^{26}\mathrm{Si}(\gamma,\mathrm{p})} + \ldots \end{aligned}\right\} \text{production} \\
& \left.\begin{aligned} & - N_{\mathrm{H}} N_{^{25}\mathrm{Al}} \langle \sigma v \rangle_{^{25}\mathrm{Al}(\mathrm{p},\gamma)} - N_{^{4}\mathrm{He}} N_{^{25}\mathrm{Al}} \langle \sigma v \rangle_{^{25}\mathrm{Al}(\alpha,\mathrm{p})} \\ & - N_{^{25}\mathrm{Al}} \lambda_{^{25}\mathrm{Al}(\beta^{+}\nu)} - N_{^{25}\mathrm{Al}} \lambda_{^{25}\mathrm{Al}(\gamma,\mathrm{p})} - \ldots \end{aligned}\right\} \text{destruction}
\end{aligned}$$

System of coupled differential equations: “nuclear reaction network”

Solved numerically

[Arnett, “Supernovae and Nucleosynthesis”, Princeton University Press, 1996]

Special Case #1: Rates for Smoothly Varying S-Factors ("non-resonant")

$$\sigma(E) \equiv \frac{1}{E} e^{-2\pi\eta} S(E)$$

$$N_A\langle\sigma v\rangle = \left(\frac{8}{\pi m_{01}}\right)^{1/2} \frac{N_A}{(kT)^{3/2}} \int_0^\infty E\,\sigma(E)\, e^{-E/kT}\, dE$$

$$= \left(\frac{8}{\pi m_{01}}\right)^{1/2} \frac{N_A}{(kT)^{3/2}} S_0 \int_0^\infty e^{-2\pi\eta}\, e^{-E/kT}\, dE$$

"Gamow peak"

Represents the energy range over which most nuclear reactions occur in a plasma!

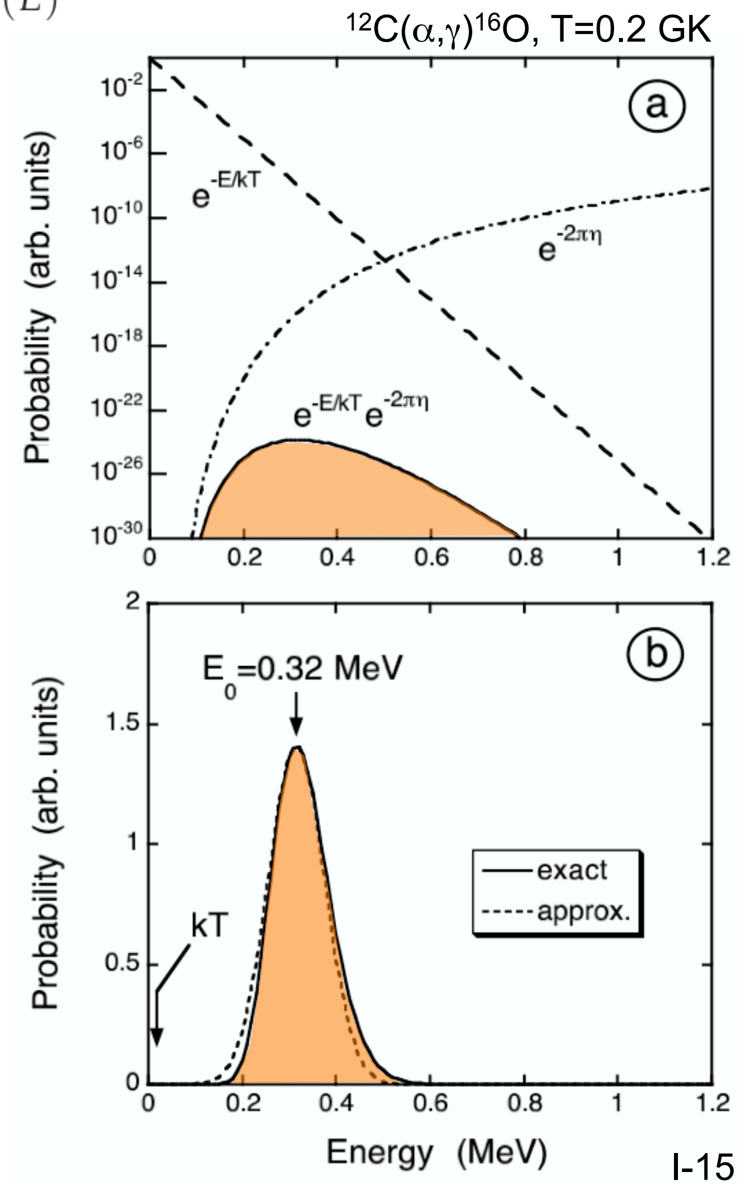
Location and 1/e width of Gamow peak:

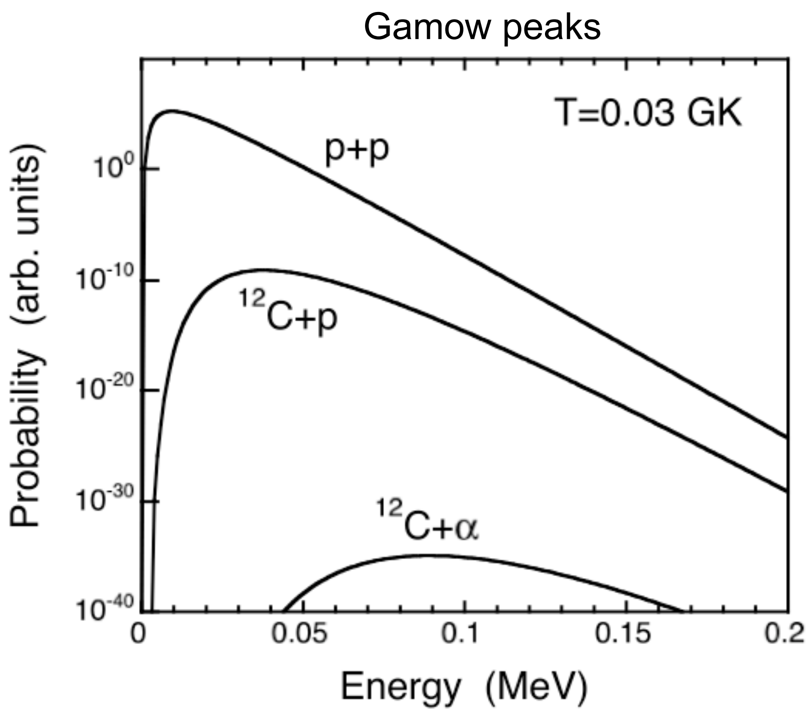
$$E_0 = \left[\left(\frac{\pi}{\hbar}\right)^2 \left(Z_0 Z_1 e^2\right)^2 \left(\frac{m_{01}}{2}\right) (kT)^2\right]^{1/3}$$

$$= 0.1220 \left(Z_0^2 Z_1^2 \frac{M_0 M_1}{M_0 + M_1} T_9^2\right)^{1/3} \qquad \text{(MeV)}$$

$$\Delta = \frac{4}{\sqrt{3}}\sqrt{E_0 kT} = 0.2368 \left(Z_0^2 Z_1^2 \frac{M_0 M_1}{M_0 + M_1} T_9^5\right)^{1/6} \qquad \text{(MeV)}$$

however, see: Newton, Iliadis et al., Phys. Rev. C 045801 (2007)



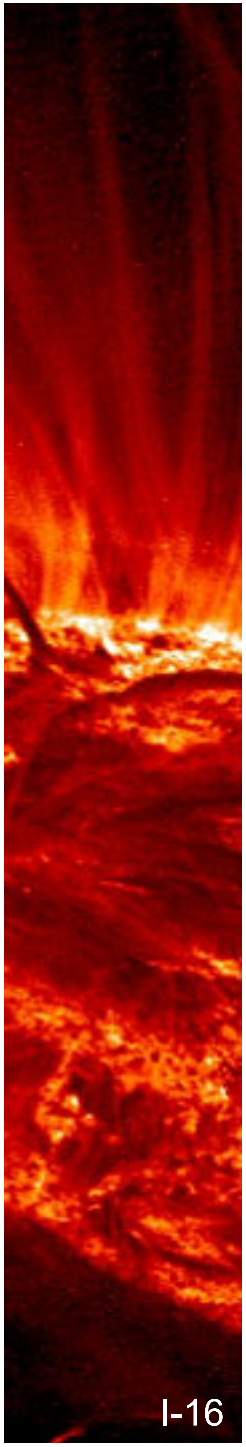


Important aspects:

(i) Gamow peak shifts to higher energy for increasing charges Z_p and Z_t

(ii) at same time, area under Gamow peak decreases drastically

Conclusion: for a mixture of different nuclei in a plasma, those reactions with the smallest Coulomb barrier produce most of the energy and are consumed most rapidly [→ stellar burning stages, see Lecture #2]



Special Case #2: Rates for “Narrow” Resonances (“Γ_i const over total Γ”)

Breit-Wigner formula (energy-**independent** partial widths)

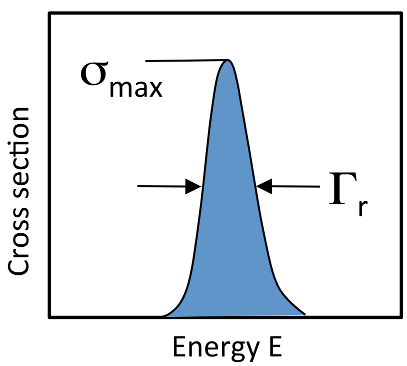
$$N_A\langle\sigma v\rangle = \left(\frac{8}{\pi m_{01}}\right)^{1/2} \frac{N_A}{(kT)^{3/2}} \int_0^\infty E\,\sigma(E)\,e^{-E/kT}\,dE$$

$$= N_A \frac{\sqrt{2\pi}\hbar^2}{(m_{01}kT)^{3/2}} e^{-E_r/kT}\,\omega\frac{\Gamma_a\Gamma_b}{\Gamma}2\pi$$

- resonance energy needs to be known rather precisely
- takes into account only rate contribution at E_r

“resonance strength” $\omega\gamma$:

- proportional to area under narrow resonance curve
- energy-dependence of σ not important



$$\omega\gamma \propto \sigma_{max}\cdot\Gamma_r$$

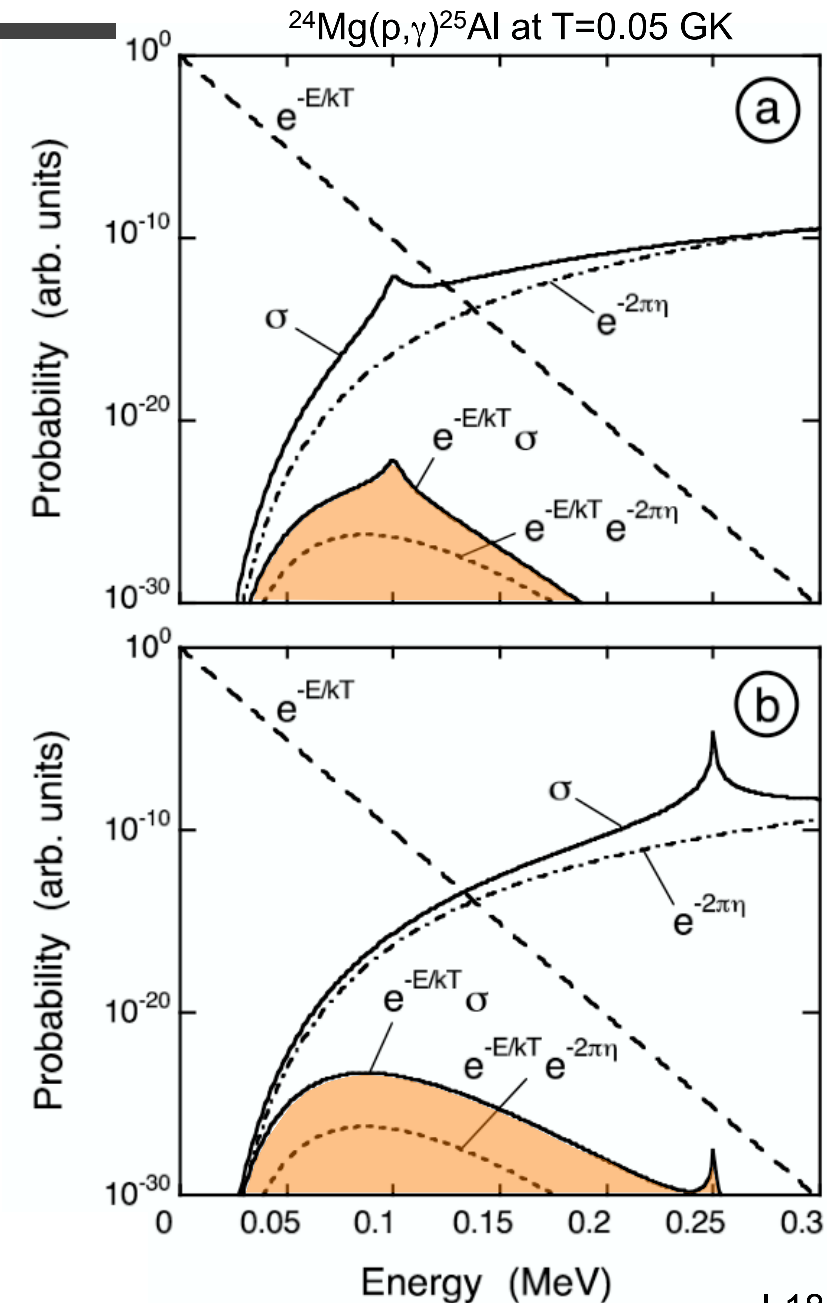
Special case #3: Rates for "Broad Resonances"

Breit-Wigner formula (energy-**dependent** partial widths)

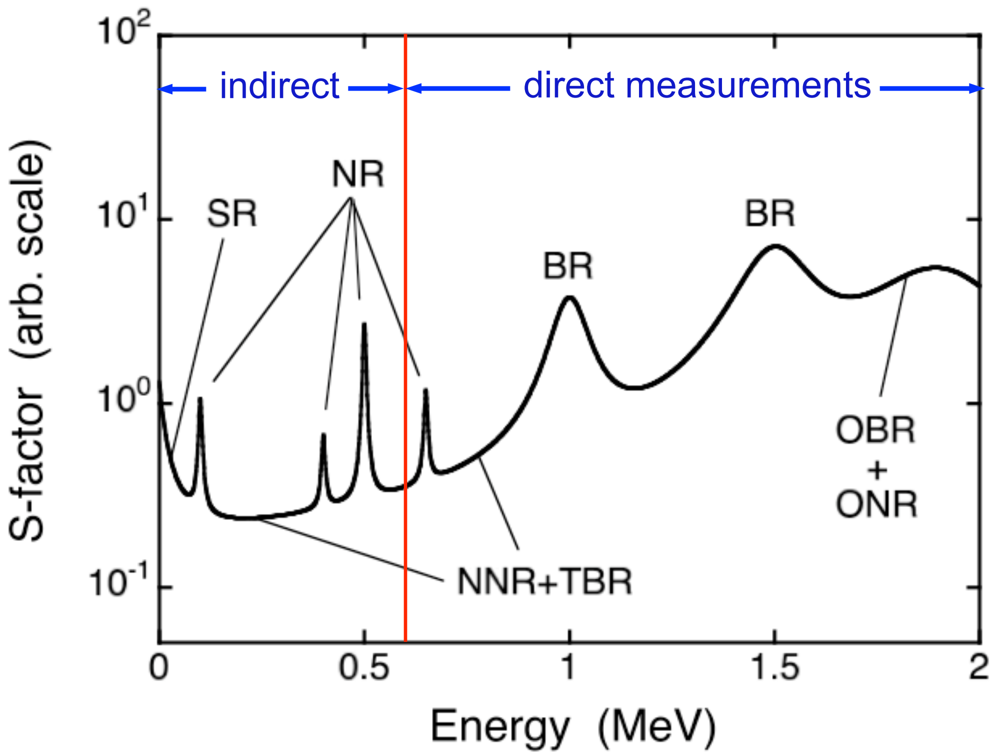
$$N_A\langle\sigma v\rangle = \left(\frac{8}{\pi m_{01}}\right)^{1/2} \frac{N_A}{(kT)^{3/2}} \int_0^\infty E\,\boxed{\sigma(E)}\,e^{-E/kT}\,dE$$

rate can be found from numerical integration

There are two contributions to the rate:
(i) from "narrow resonance" at E_r
(ii) from tail of broad resonance



Total Thermonuclear Reaction Rate



Need to consider:

- non-resonant processes
- narrow resonances
- broad resonances
- subthreshold resonances
- interferences
- continuum

every nuclear reaction represents a special case !

Lecture #2: Nuclear Burning Stages [excl. explosive burning]

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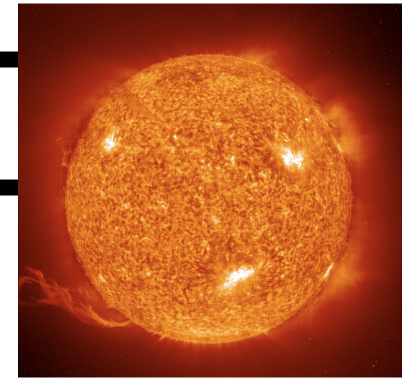
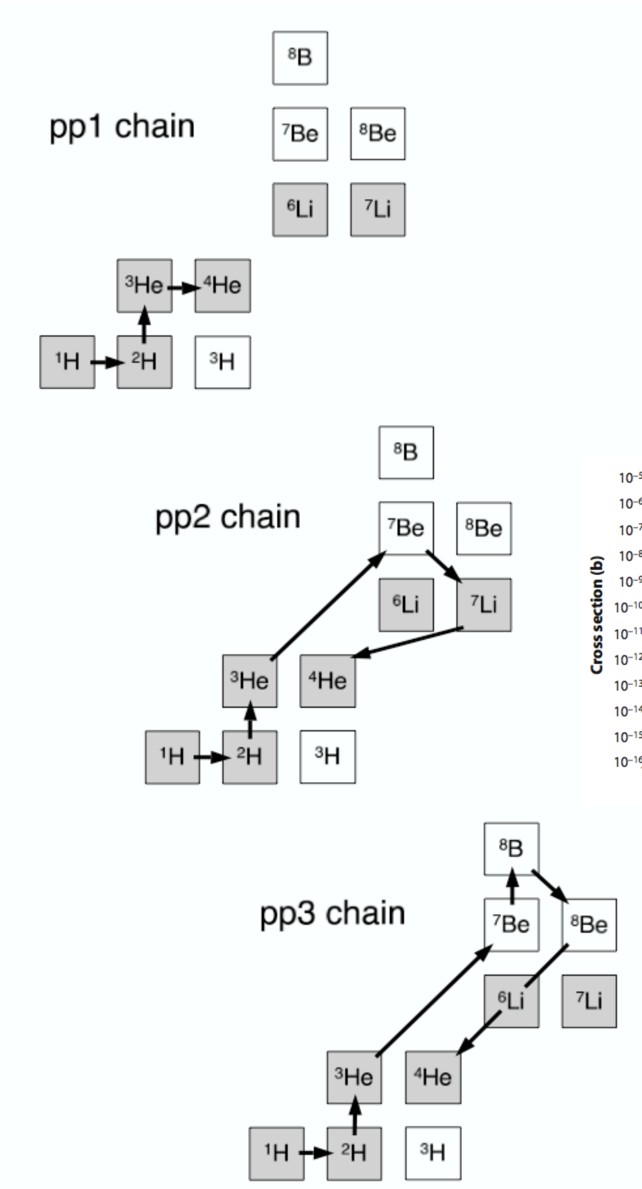
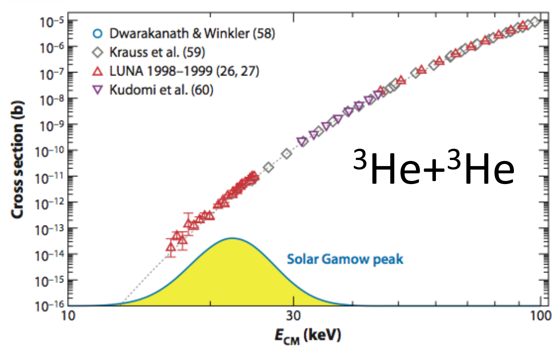
Hydrostatic Hydrogen Burning: sun (T=15.6 MK), stellar core (T=8-55 MK), shell of AGB stars (T=45-100 MK)



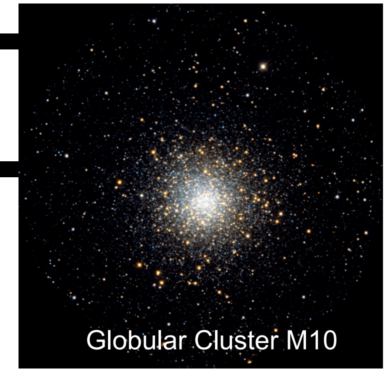


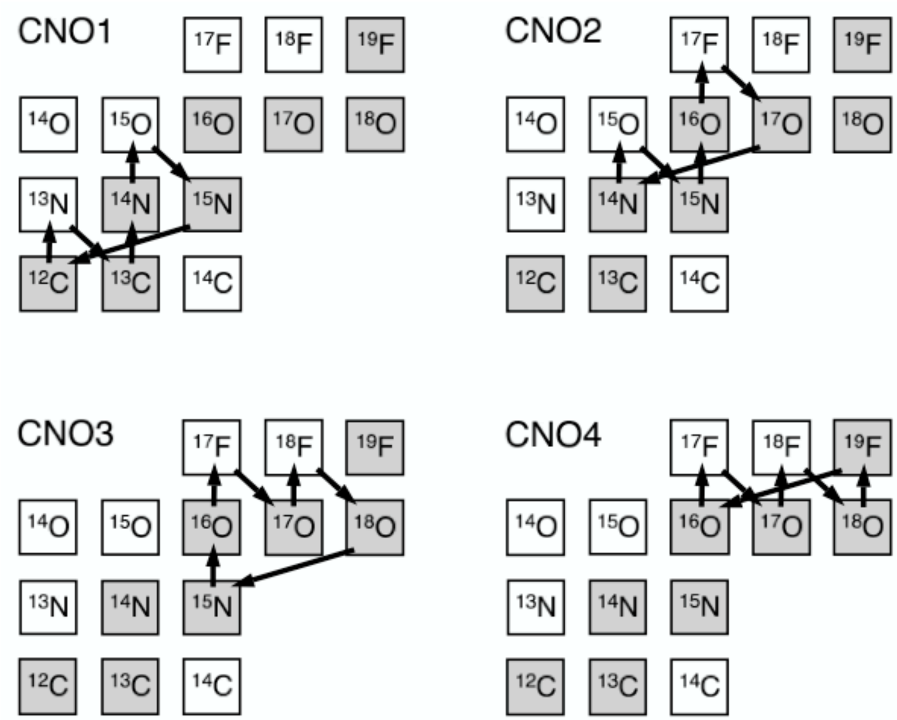
pp1 chain	pp2 chain	pp3 chain
$p(p,e^+\nu)d$	$p(p,e^+\nu)d$	$p(p,e^+\nu)d$
$d(p,\gamma)^3He$	$d(p,\gamma)^3He$	$d(p,\gamma)^3He$
$^3He(^3He,2p)\alpha$	$^3He(\alpha,\gamma)^7Be$	$^3He(\alpha,\gamma)^7Be$
	$^7Be(e^-,\nu)^7Li$	$^7Be(p,\gamma)^8B$
	$^7Li(p,\alpha)\alpha$	$^8B(\beta^+\nu)^8Be$
		$^8Be(\alpha)\alpha$



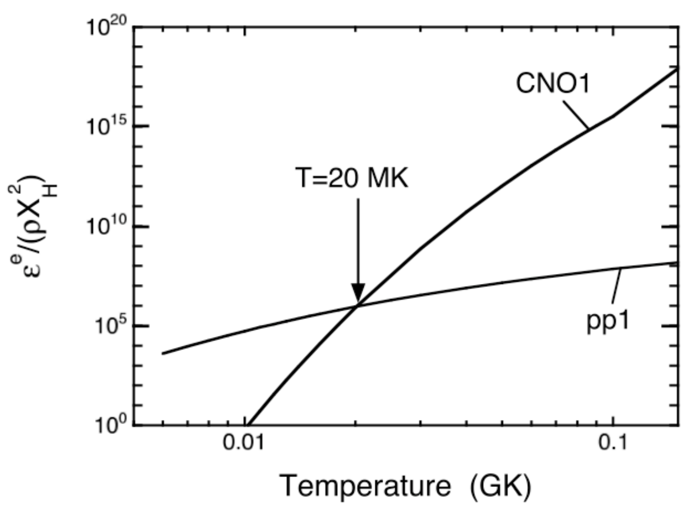
- 4H→^{4}He releases 26.7 MeV
- reactions are non-resonant at low energies
- p+p [slowest reaction] has not been measured
- d+p, ^{3}He+^{3}He, ^{3}He+α have been measured by LUNA collaboration
- 90% of Sun's energy produced by pp1 chain

Hydrostatic Hydrogen Burning: sun (T=15.6 MK), stellar core (T=8-55 MK), shell of AGB stars (T=45-100 MK)





- ^{12}C and ^{16}O nuclei act as catalysts
- branchings: (p,α) stronger than (p,γ)
- ^{14}N(p,γ)^{15}O slowest reaction in CNO1
  has been measured by LUNA/LENA
- solar: ^{13}C/^{12}C=0.01;
  CNO1: ^{13}C/^{12}C=0.25 (“steady state”)
- T>20 MK: CNO1 faster than pp1



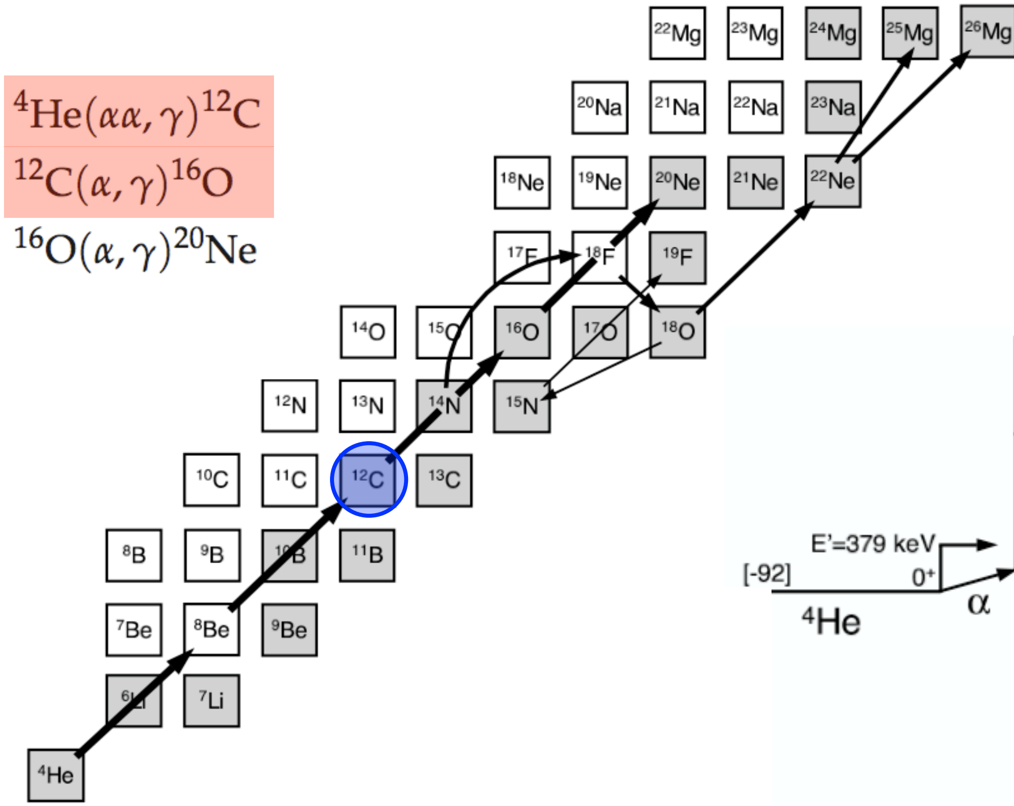
CNO1	CNO2	CNO3	CNO4
^{12}C(p,γ)^{13}N	^{14}N(p,γ)^{15}O	^{15}N(p,γ)^{16}O	^{16}O(p,γ)^{17}F
^{13}N($\beta^+\nu$)^{13}C	^{15}O($\beta^+\nu$)^{15}N	^{16}O(p,γ)^{17}F	^{17}F($\beta^+\nu$)^{17}O
^{13}C(p,γ)^{14}N	^{15}N(p,γ)^{16}O	^{17}F($\beta^+\nu$)^{17}O	^{17}O(p,γ)^{18}F
^{14}N(p,γ)^{15}O	^{16}O(p,γ)^{17}F	^{17}O(p,γ)^{18}F	^{18}F($\beta^+\nu$)^{18}O
^{15}O($\beta^+\nu$)^{15}N	^{17}F($\beta^+\nu$)^{17}O	^{18}F($\beta^+\nu$)^{18}O	^{18}O(p,γ)^{19}F
^{15}N(p,α)^{12}C	^{17}O(p,α)^{14}N	^{18}O(p,α)^{15}N	^{19}F(p,α)^{16}O

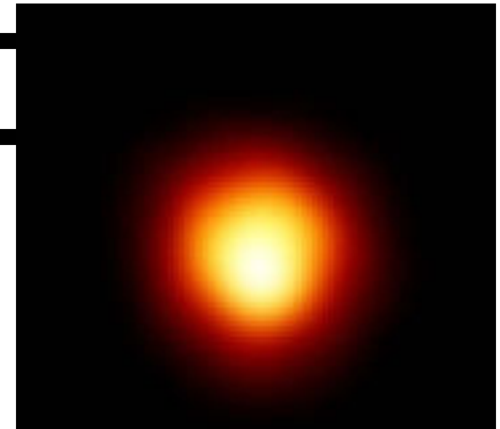
Helium Burning: massive stars (T=100-400 MK)

$^{4}He(\alpha\alpha,\gamma)^{12}C$

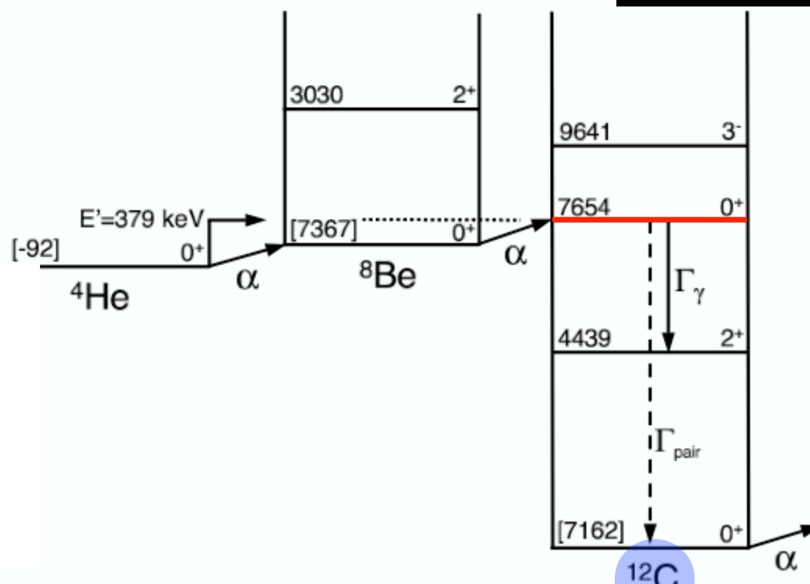
$^{12}C(\alpha,\gamma)^{16}O$

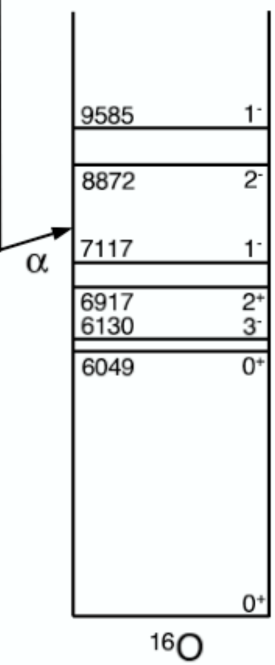
$^{16}O(\alpha,\gamma)^{20}Ne$





Betelgeuse (α Orionis)

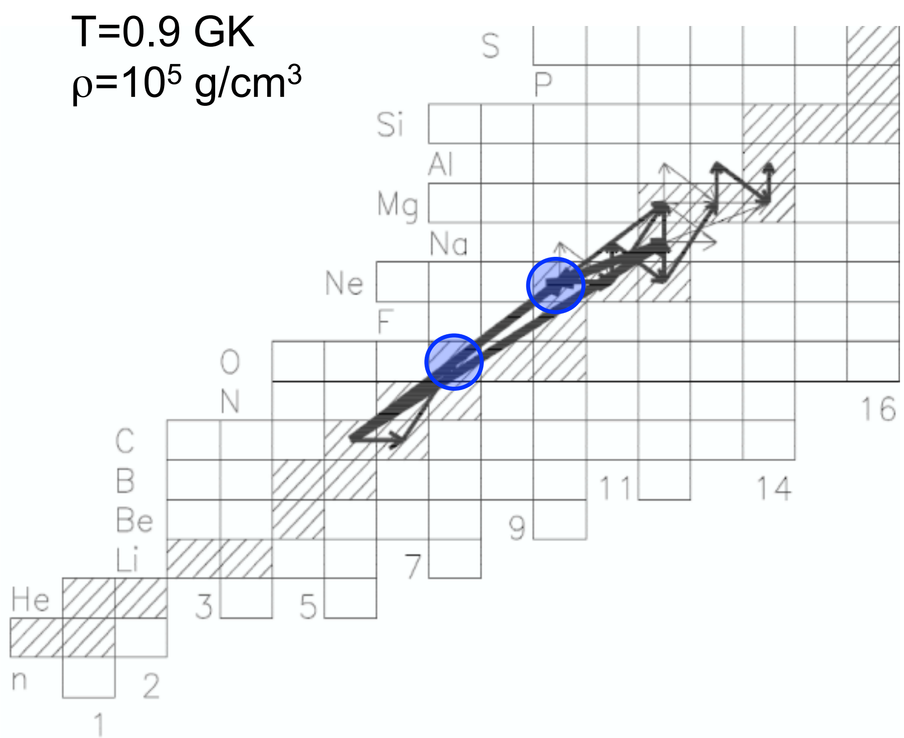


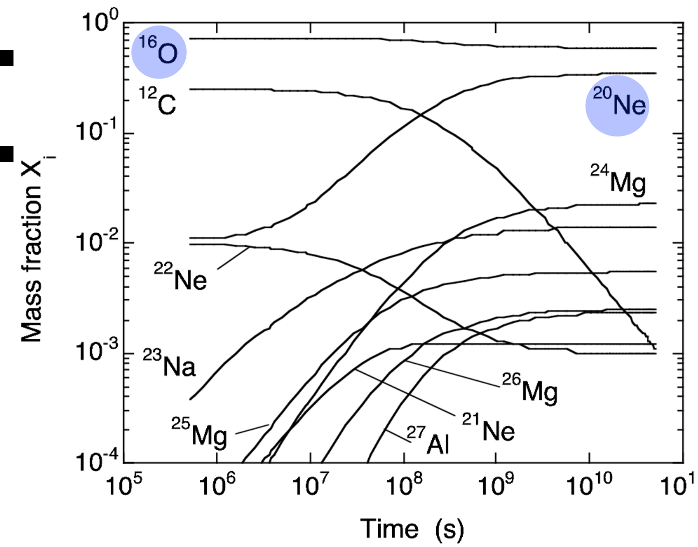


- 3α reaction cannot be measured directly (±15%)
- $^{12}C(\alpha,\gamma)^{16}O$ slow, crucial reaction [determines $^{12}C/^{16}O$ ratio]
- $^{16}O(\alpha,\gamma)^{20}Ne$ very slow
- ashes: ^{12}C, ^{16}O
- last core burning stage for evolution of low-mass stars [Sun]; they eventually become “CO White Dwarfs”

Fred Hoyle (1915-2001)

Carbon Burning: core (T=0.6-1.0 GK)



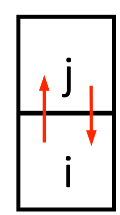


- Primary reactions:
  $^{12}C(^{12}C,p)^{23}Na$
  $^{12}C(^{12}C,\alpha)^{20}Ne$
  $^{12}C(^{12}C,n)^{23}Mg$
  + several secondary reactions
- ashes: ^{16}O, ^{20}Ne
- last core burning stage for evolution of intermediate-mass stars; they eventually become "ONe White Dwarfs"

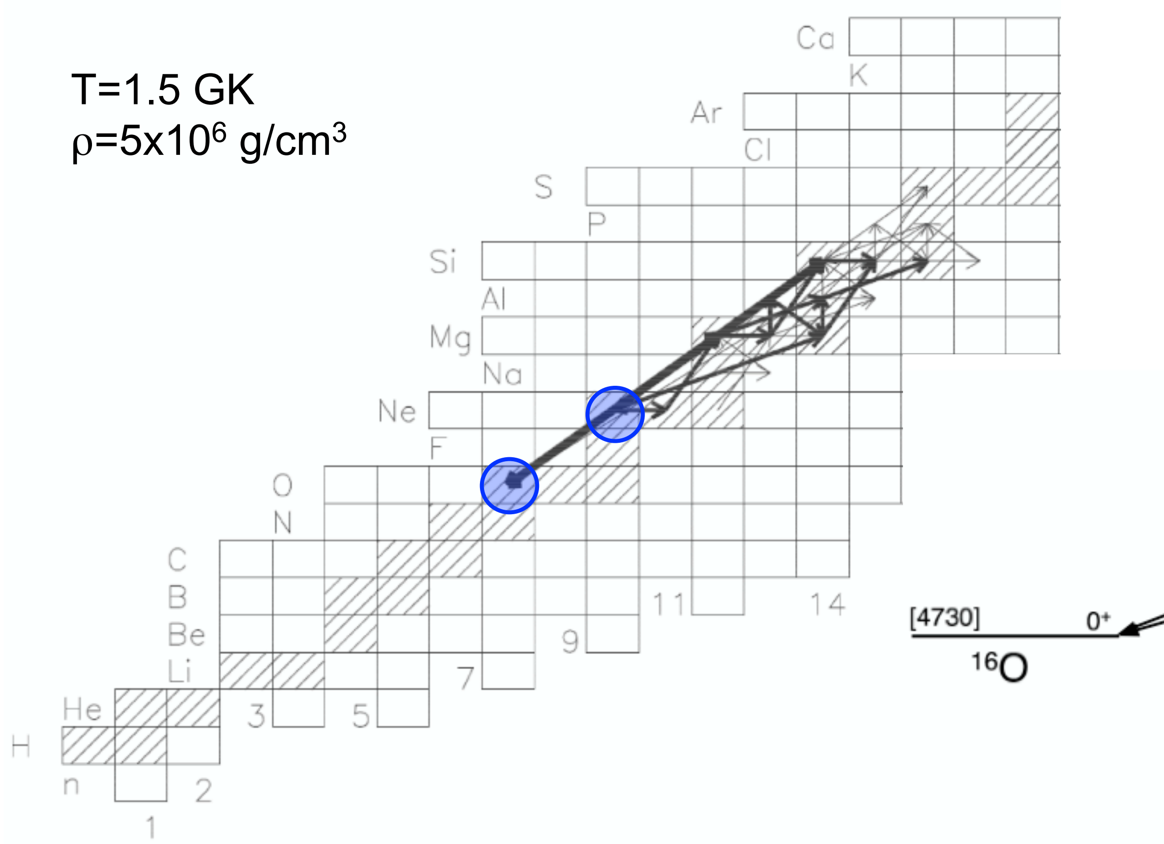
"Abundance flows"

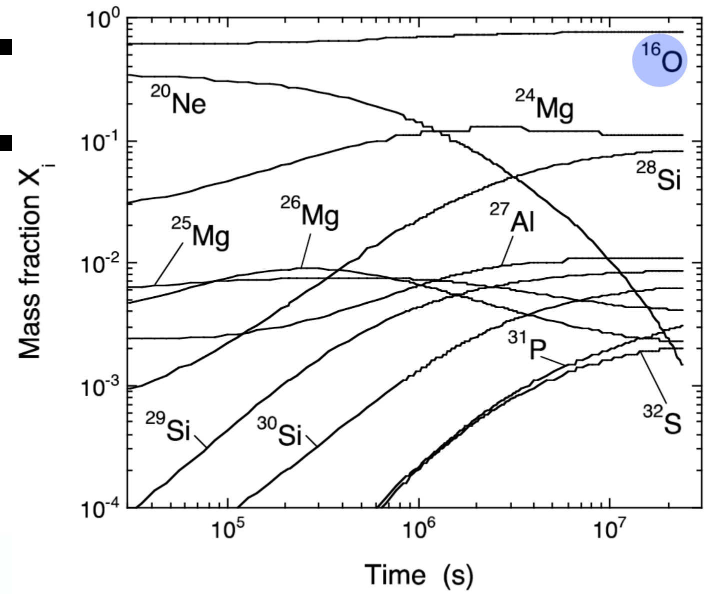
$$F_{ij} = \int f_{ij}\,dt = \int \left[\left(\frac{dN_i}{dt} \right)_{i\to j} - \left(\frac{dN_j}{dt} \right)_{j\to i} \right] dt$$

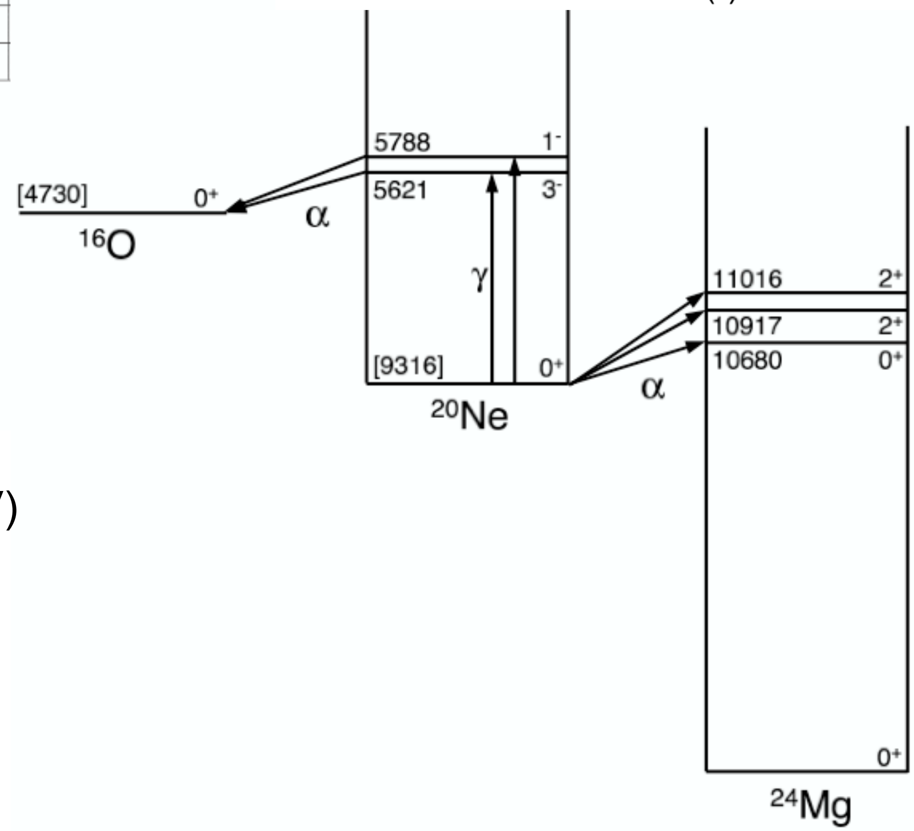
"time-integrated net abundance flow"



Neon Burning: core (T=1.2-1.8 GK)



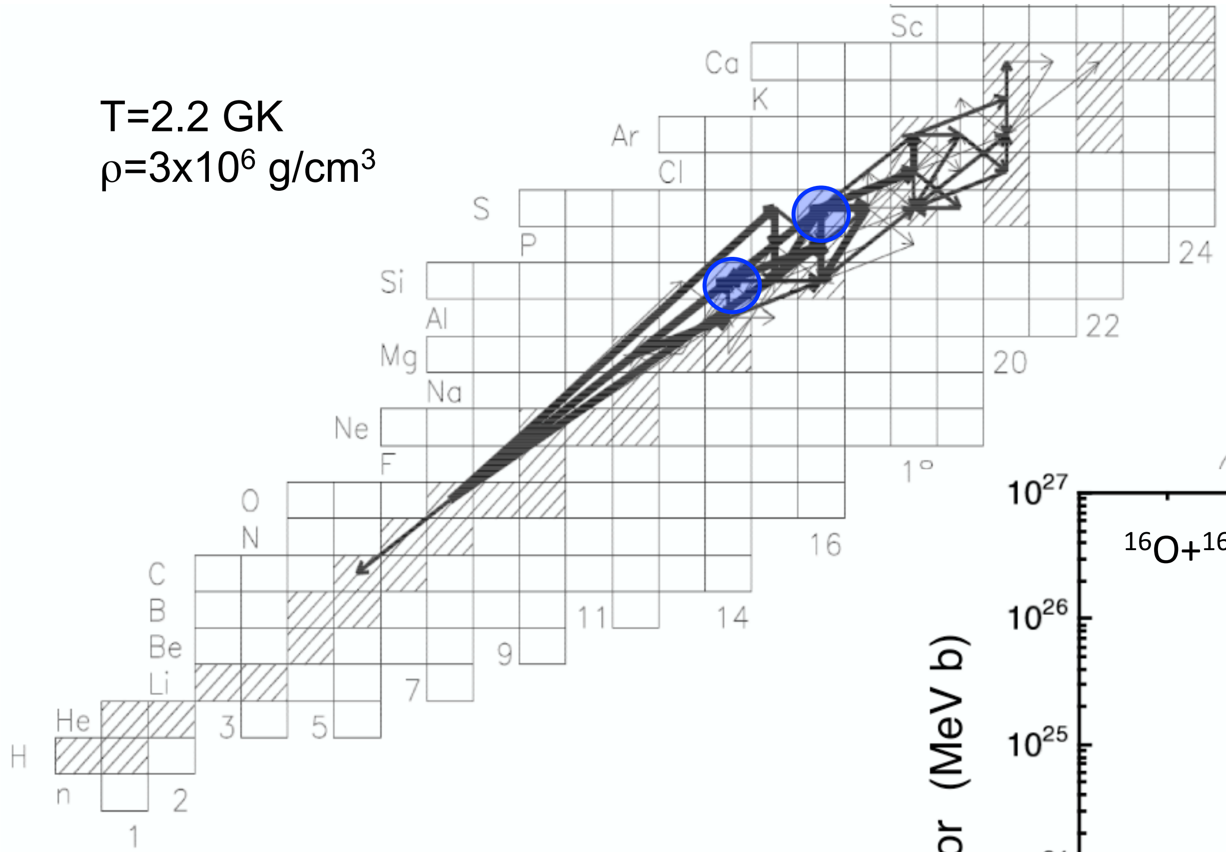


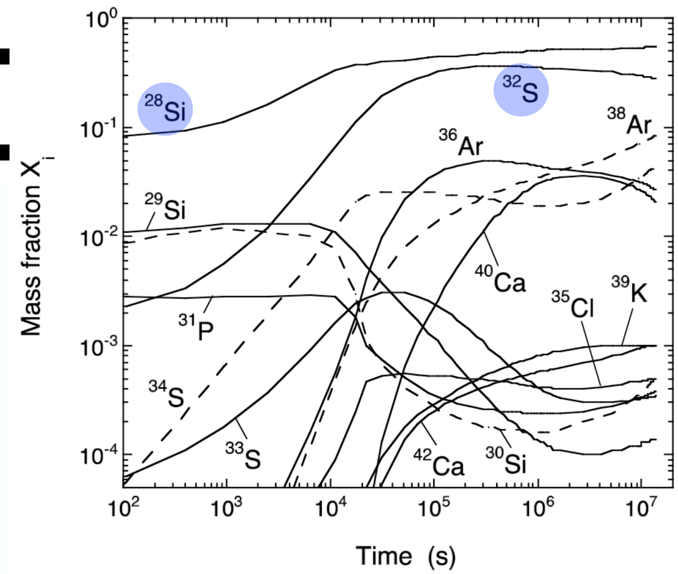


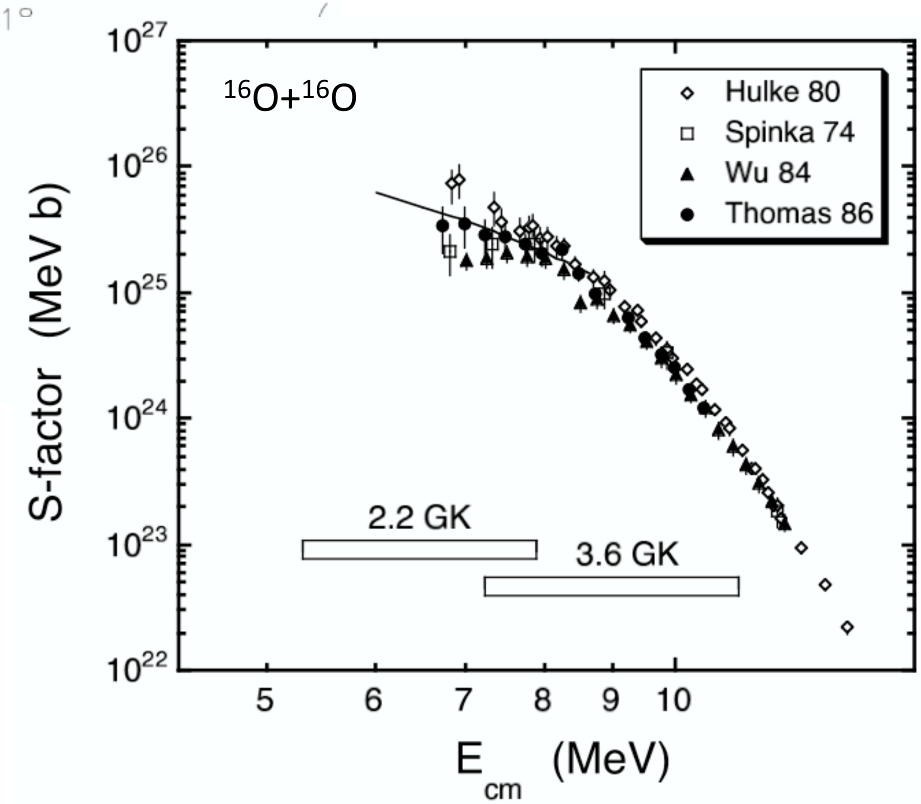
- Primary reaction:
  $^{20}Ne(\gamma,\alpha)^{16}O$ (Q=-4730 keV)
- Secondary reactions
  $^{20}Ne(\alpha,\gamma)^{24}Mg(\alpha,\gamma)^{28}Si$
  + more
- ashes: ^{16}O

Oxygen Burning: core (T=1.5-2.7 GK)

T=2.2 GK
$\rho=3 \times 10^6$ g/cm^3



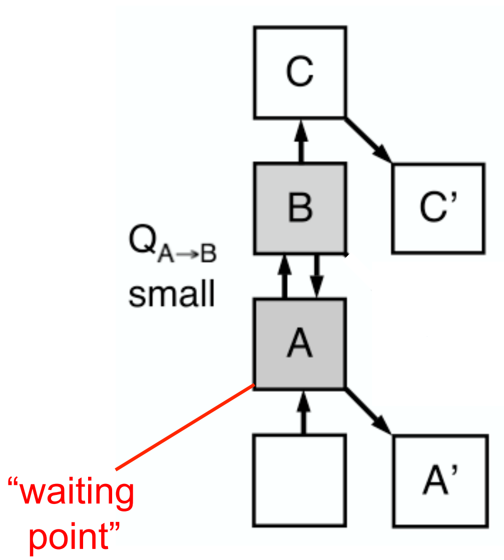


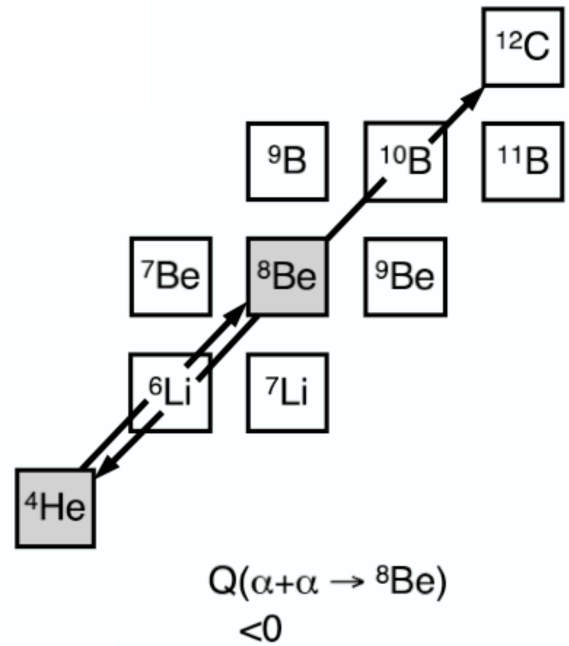


- Primary reactions:
  $^{16}O(^{16}O,p)^{31}P$
  $^{16}O(^{16}O,\alpha)^{28}Si$
  ...
  + several secondary reactions
- ashes: ^{28}Si, ^{32}S

Reaction Rate Equilibria: r = $r_{A\to B} - r_{B\to A} = 0$

$$\lambda_1(0) = \rho \frac{X_1}{M_1} N_A \langle \sigma v \rangle_{01}$$





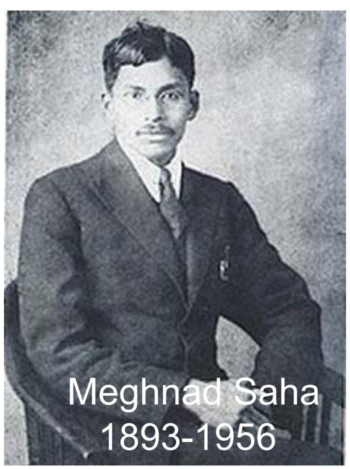
From Saha statistical equation and reciprocity theorem:

$$\lambda_{A\to B\to C} = \frac{\lambda_{A\to B}}{\lambda_{B\to A}} \lambda_{B\to C}$$

$$= N_a \left(\frac{h^2}{2\pi}\right)^{3/2} \frac{1}{(m_{Aa}kT)^{3/2}} \frac{(2j_B+1)}{(2j_A+1)(2j_a+1)}$$

$$\times \frac{G_B^{\mathrm{norm}}}{G_A^{\mathrm{norm}} G_a^{\mathrm{norm}}} e^{Q_{A\to B}/kT} \lambda_{B\to C}$$

independent of reaction rate for A→B!



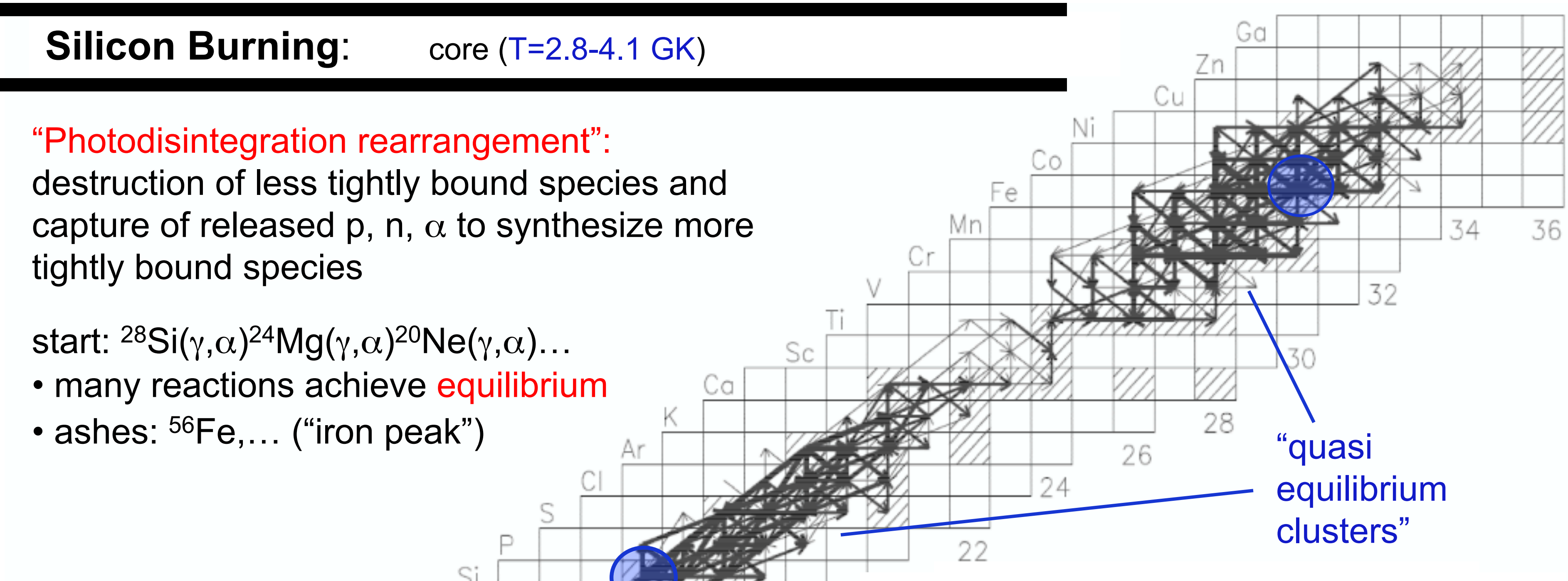
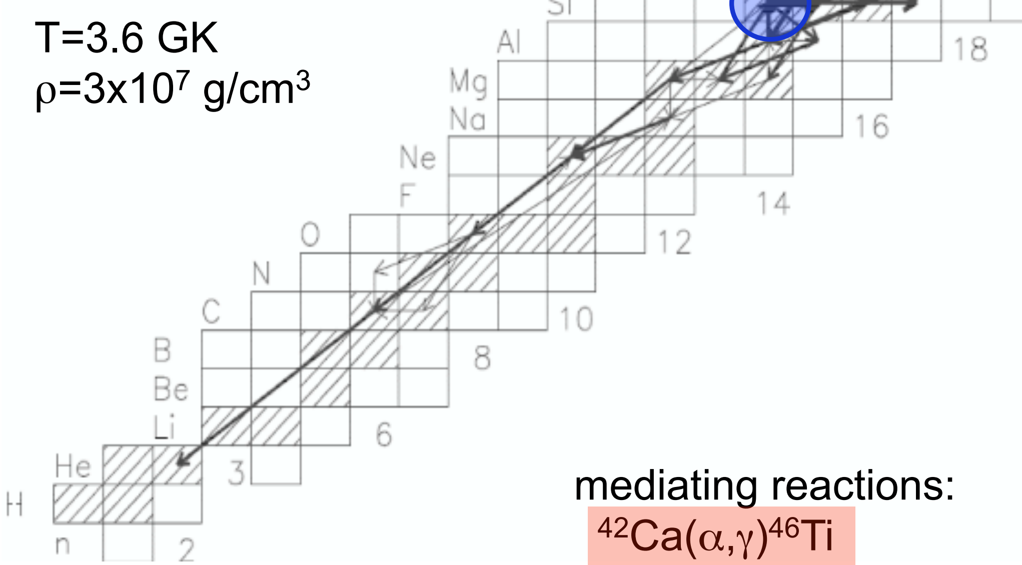
$$\lambda_{3\alpha} = 0.23673 \frac{(\rho X_\alpha)^2}{T_9^3} e^{-11.6048E'/T_9} \omega\gamma_{^8\mathrm{Be}(\alpha,\gamma)} \qquad (\mathrm{s}^{-1})$$

Silicon Burning: core (T=2.8-4.1 GK)

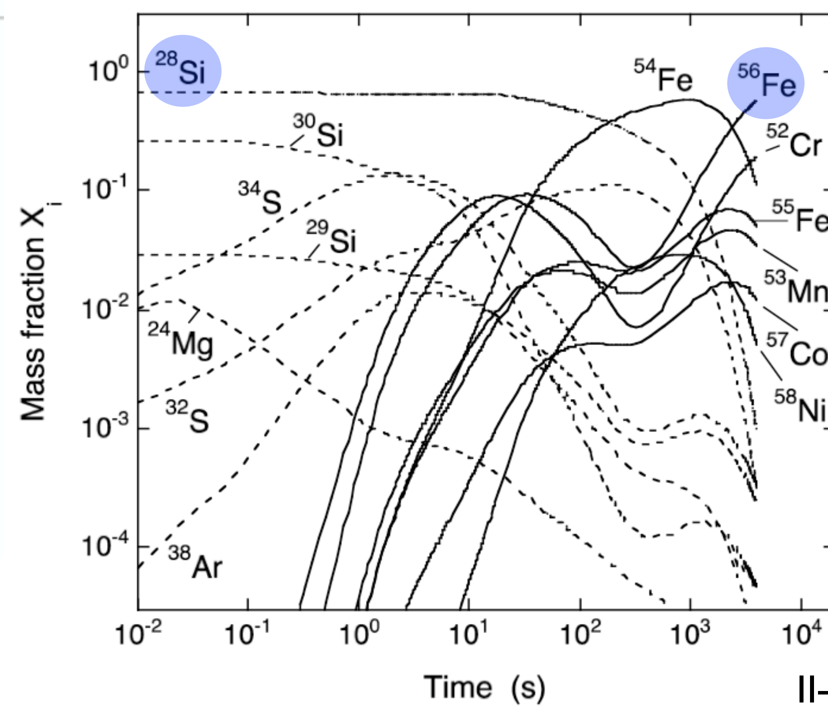
"Photodisintegration rearrangement":
destruction of less tightly bound species and capture of released p, n, α to synthesize more tightly bound species

start: $^{28}Si(\gamma,\alpha)^{24}Mg(\gamma,\alpha)^{20}Ne(\gamma,\alpha)$…
- many reactions achieve equilibrium
- ashes: ^{56}Fe,… ("iron peak")

T=3.6 GK
$\rho=3x10^{7}$ g/cm^{3}

mediating reactions:
$^{42}Ca(\alpha,\gamma)^{46}Ti$
$^{41}K(\alpha,p)^{44}Ca$
$^{45}Sc(p,\gamma)^{46}Ti$



Nuclear Statistical Equilibrium I: General Description

As ^{28}Si disappears in the core at the end of Si burning, T increases, until all non-equilibrated reactions come into equilibrium [last reaction: 3α reaction]

One large equilibrium cluster stretches from p, n, α to Fe peak:
"Nuclear Statistical Equilibrium" (NSE)

Abundance of each nuclide can be calculated from repeated application of Saha equation:

$$\text{For species } {}^{A}_{\pi}Y_{\nu}: \quad N_Y = N_\mathrm{p}^{\pi} N_\mathrm{n}^{\nu} \frac{1}{\theta^{A-1}} \left(\frac{M_Y}{M_\mathrm{p}^{\pi} M_\mathrm{n}^{\nu}} \right)^{3/2} \frac{g_Y}{2^A} G_Y^{\mathrm{norm}} e^{B(Y)/kT}$$

In NSE, abundance of any nuclide is determined by: temperature, density, neutron excess

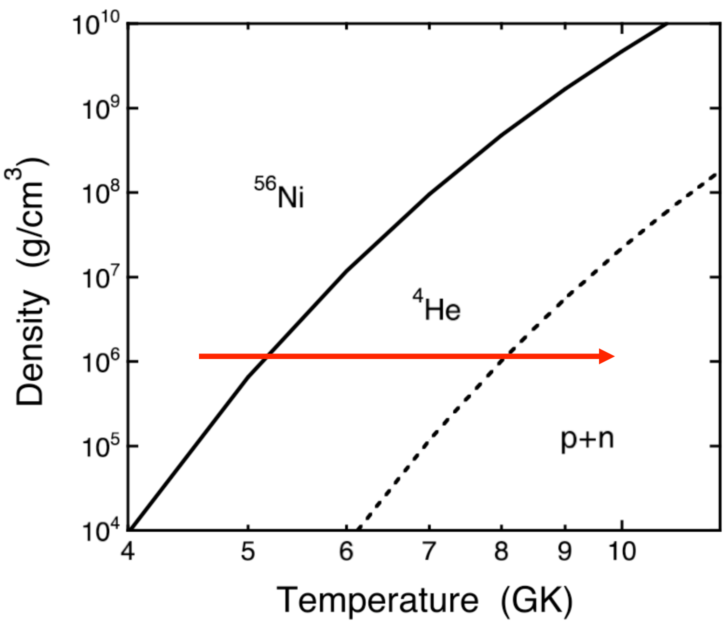
$$\eta \equiv \sum_i \frac{(N_i - Z_i)}{M_i} X_i$$

N_i, Z_i, M_i : number of n, p; atomic mass

M_i, X_i : atomic mass, mass fraction

Represents number of excess neutrons per nucleon (can only change as result of weak interactions!)

Nuclear Statistical Equilibrium II: Interesting Properties



Assume first that $\eta=0$ when NSE is established and Si burning has mainly produced ^{56}Ni (N=Z=28) in the Fe peak besides ^{4}He, p, n…

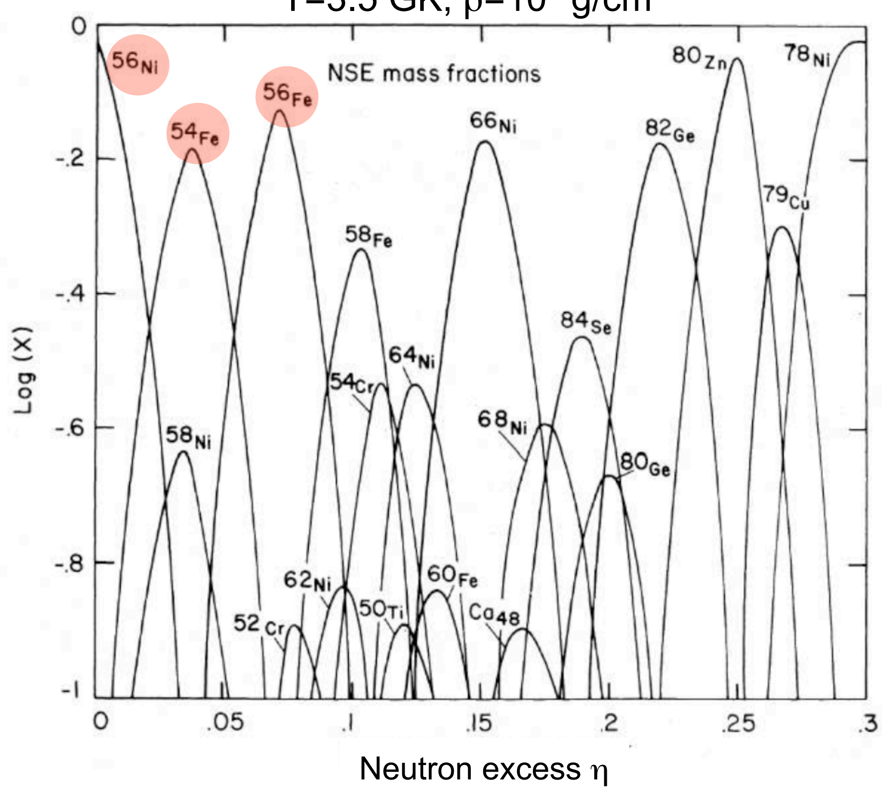
At ρ=const and T rising: increasing fraction of composition resides in light particles (p, n, α)

Dominant species:

^{56}Ni for $\eta=0$ (N-Z)/M=(28-28)/56=0
^{54}Fe for $\eta=0.04$ (N-Z)/M=(28-26)/54=0.04
^{56}Fe for $\eta=0.07$ (N-Z)/M=(30-26)/56=0.07
…

η needs to be monitored very carefully at each of the previous burning stages!
[stellar weak interaction rates need to be known]

T=3.5 GK, $\rho=10^7$ g/cm^3



Hartmann, Woosley & El Eid, ApJ 297, 837 (1985)